ELSEVIER
Journal of Geometry and Physics 24 (1998) 164–172
JOURNAL OF GEOMETRY AND PHYSICS

On tangential properties of the Gutt $*$-product

Santos Asin Lares [1]
Mathematical Institute, University of Warwick, Mathematical Institute, Conventry, CV4 7AL, UK

Received 23 September 1996

Abstract

We establish necessary and sufficient conditions on a Lie algebra $\mathbf{g}$, under which the Gutt $*$-product on $\mathbf{g}^*$ is tangential to a given coadjoint orbit.

Subj. Class.: Differential geometry
1991 MSC: 17B65, 16A24, 58F05
Keywords: Gutt $*$-product; Coadjoint orbit

1. Introduction

In the late 1970s, the concept of a *star-product* on a Poisson manifold was introduced by Bayen et al. [3]. That is, to define an associative multiplication operation $*$ (depending on a parameter $\lambda \in \mathbb{C}$) of two functions, so that the space of smooth functions with this $*$-product as a multiplication operation would be a *formal deformation* of the commutative algebra.

The general question of the existence of such a product for symplectic manifolds has been completely solved by several authors, using various techniques [7,9,13], and even for some special Poisson manifolds [9,12,14].

A crucial point in the study of the existence of $*$-products on Poisson manifolds is the fact that every Poisson manifold splits into a collection of symplectic submanifolds [16], known as *the leaves of the symplectic foliation*. One naturally asks whether a $*$-product can be constructed on a Poisson manifold by *'gluing together smoothly'* the star-products defined on the symplectic leaves. Such $*$-products are called *tangential* [5], and their existence has

[1] Supported by a grant from CONACYT (Mexico).

PII S0393-0440(96)00038-1

been proved for *regular Poisson manifolds* [9,12]. However, for arbitrary Poisson manifolds it is an open problem.

Therefore, it is natural to start considering a Poisson manifold which is endowed with a simple Poisson bracket, and this can be the dual of a Lie algebra **g** (with the Lie Poisson bracket). These Poisson manifolds are not regular (unless **g** is abelian).

The existence of a $*$-product for $\mathbf{g}^*$ has been shown by Gutt [11]. In fact, in [11], Gutt constructed a $*$-product (*Gutt $*$-product*) on the symplectic manifold $T^*\mathbf{G}$ of any Lie group **G**, and the 'vertical' part of this $*$-product is a $*$-product on $\mathbf{g}^*$. Unfortunately it is not in general *tangential*.

Recently, Cahen et al. [5] have proved that in the case of a semisimple Lie algebra there are no *differential* and tangential (to all the orbits) $*$-products on $\mathbf{g}^*$.

This work aims to establish necessary and sufficient conditions on a Lie algebra **g**, under which the Gutt $*$-product gives rise to a tangential $*$-product on a given orbit.

2. Definitions and notation

Let **G** be a real, connected Lie group with Lie algebra **g**. (Throughout the exposition, **g** will always be real and finite-dimensional.) Let $\mathbf{g}^*$ be the dual space of **g**. If $x \in \mathbf{g}^*$ and $g \in \mathbf{G}$, the *coadjoint representation* is defined by

$$\langle Ad_g^* x, Y\rangle = \langle x, Ad_{g^{-1}} Y\rangle \quad \forall Y \in \mathbf{g},$$

where Ad_g stands for the *adjoint representation*. Similarly, there is a linear representation of **g** in $\mathbf{g}^*$, that is, if $X \in \mathbf{g}$ then $X \cdot x$ is defined by

$$\langle ad_X^* x, Y\rangle = -\langle x, ad_X Y\rangle = -\langle x, [X, Y]\rangle.$$

Let **h** be a Lie subalgebra of **g**. By $\mathbf{h}^\perp$ we denote the annihilator of **h** in $\mathbf{g}^*$, i.e. $\mathbf{h}^\perp = \{x \in \mathbf{g}^*: x(X) = 0, \forall X \in \mathbf{h}\}$. The isotropy subgroup of $x \in \mathbf{g}^*$ in **G** is given by $\mathbf{G}_x = \{g \in \mathbf{G}: Ad_g^* x = x\}$, its Lie algebra by $\mathbf{g}_x = \{X \in \mathbf{g}: ad_X^* x = 0\}$, and the coadjoint orbit through x by O_x.

Let us recall two propositions which will be useful later on.

Proposition 1. *Let* **g** *be a Lie algebra, and* $x \in \mathbf{g}^*$. *If* $\mathbf{g}_x$ *is an ideal of* **g**, *then* $O_x \subseteq x + {\mathbf{g}_x}^\perp$. *Furthermore,* O_x *is an open set in* $x + {\mathbf{g}_x}^\perp$.

Proof. The proof follows the same pattern as in [6]. Let U_1 be an open neighborhood of $0 \in \mathbf{g}$ such that the exponential map on it is a diffeomorphism, and let $U_2 = \exp(U_1)$.

Let $X \in U_1$, $g \in U_2$ be such that $\exp(X) = g$, and let $Y \in \mathbf{g}_x$. Since,

$$\langle Ad_g^* x, Y\rangle = \langle Ad_{\exp(X)}^* x, Y\rangle = \langle x, Ad_{\exp(-X)}, Y\rangle = \langle x, \mathrm{e}^{-adX}, Y\rangle,$$

and $\mathbf{g}_x$ is an ideal, it follows that

$$\langle x, \mathrm{e}^{-adX}, Y\rangle = \langle x, Y\rangle,$$

so, $Ad_g^* x - x \in {\mathbf{g}_x}^\perp$, i.e. $Ad_g^* x \in x + {\mathbf{g}_x}^\perp$.

The rest of the proof follows directly from the fact that $\mathbf{G}$ is connected, so, it is generated by an open neighborhood of the identity, thus, we can apply the previous argument to the factors (in a neighborhood of the identity) of an arbitrary $g \in \mathbf{G}$. □

Proposition 2 [1]. *Let* $\mathbf{M}$ *be a smooth manifold. If* $\Phi : \mathbf{G} \times \mathbf{M} \to \mathbf{M}$ *is an action and* $x \in \mathbf{M}$, *then* $\tilde{\Phi} : \mathbf{G}/\mathbf{G}_x \to O_x \subset \mathbf{M}$, *given by*

$$g \cdot \mathbf{G}_x \longrightarrow \Phi_g \cdot x,$$

is an injective immersion.

The dual $\mathbf{g}^*$ of any Lie algebra $\mathbf{g}$ can be endowed with a natural *Poisson structure*, the so-called *Lie Poisson structure*. If $f, g \in C^\infty(\mathbf{g}^*)$, then for every point $x \in \mathbf{g}^*$, $\mathrm{d}f(x)$ and $\mathrm{d}g(x)$ are two linear forms on $\mathbf{g}^*$, that we may consider as elements of $\mathbf{g}$. The Poisson bracket $\{f, g\}$ is defined by

$$\{f, g\}(x) = \langle x, [\mathrm{d}f(x), \mathrm{d}g(x)]\rangle.$$

In the Poisson manifold $\mathbf{g}^*$, the orbits of the coadjoint action are precisely the leaves of the symplectic foliation induced by the Lie Poisson bracket.

2.1. $*$-Products

Now, we will recall the definition of a $*$-product, the *Hochschild cohomology*, and some other basic facts concerning $*$-products.

Let $(\mathbf{M}, \Lambda)$ be a Poisson manifold. The space $N = C^\infty(\mathbf{M})$ admits two algebraic structures, a structure of an *associative algebra* given by the usual product of the functions and a structure of a *Lie algebra* given by the Poisson bracket.

Let $N[[\lambda]]$ be the space of formal power series in a parameter $\lambda \in \mathbb{C}$, with coefficients in N.

Definition 1 [3]. A $*$-product on $(\mathbf{M}, \Lambda)$ is a bilinear map $N^2 \longrightarrow N[[\lambda]]$ defined by

$$(f, g) \longrightarrow f * g = \sum_{n=0}^{\infty} \lambda^n C_n(f, g),$$

where the so-called *cochains* C_n are bilinear maps with values in N and satisfy the following axioms:

1. $C_0(f, g) = fg,\ C_1(f, g) = \{f, g\},\ \forall f, g \in C^\infty(\mathbf{M})$,
2. $C_n(f, g) = (-1)^n C_n(g, f),\ \forall f, g \in C^\infty(\mathbf{M}),\ \forall n \in \mathbb{N}$,
3. $C_n(f, k) = 0,\ \forall f \in C^\infty(\mathbf{M}),\ \forall k \in \mathbb{R},\ \forall n \geq 1$,
4. $\sum_{r+s=k} C_r(C_s(f, g), h) = \sum_{r+s=k} C_r(f, C_s, (g, h)), k \geq 0$.

The theory of *deformations* in the sense of [10] relates the deformations of an associative algebra to the corresponding *Hochschild cohomology*.

Definition 2. A p-cochain C is a p-linear map $N^p \to N$. The *coboundary* of a p-cochain is the $p+1$-cochain ∂C given by

$$\begin{aligned}\partial C(u_0, \dots, u_p) = u_0 C(u_1, \dots, u_p) - C(u_0 u_1, u_2, \dots, u_p) + C(u_0, u_1 u_2, \dots, u_p) \\ + \cdots + (-1)^p C(u_0, u_1, \dots, u_{p-1} u_p) \\ + (-1)^{p+1} C(u_0, \dots, u_{p-1}) u_p.\end{aligned}$$

The pth Hochschild cohomology group is denoted by $H^p_{\mathrm{diff}}(N)$. The subscript 'diff' indicates that all the cochains considered are multidifferential operators which vanish on the constants.

A cochain C is said to be *differential* if it is given by differential operators on each argument.

A $*$-product is said to be differential if all the C_n are differential cochains.

2.2. Gutt $*$-product

Let $\mathbf{g}$ be a Lie algebra. The symmetric algebra $S(\mathbf{g})$ over $\mathbf{g}$, is naturally identified with the algebra of real-valued polynomials on the dual $\mathbf{g}^*$. Let $S^k(\mathbf{g})$ be the space of homogeneous polynomials of degree k.

Gutt in [11] has constructed a $*$-product on $S(\mathbf{g})$, which we may summarize as follows.

Let $U(\mathbf{g})$ be the *universal enveloping algebra* of $\mathbf{g}$, and let $\otimes$ denote the product in $U(\mathbf{g})$. Let $\sigma : S(\mathbf{g}) \to U(\mathbf{g})$ be the linear bijection defined by symmetrization, i.e.

$$\sigma(X_{i_1}, \dots, X_{i_p}) = \frac{1}{p!} \sum\nolimits_{s \in S_p} X_{i_{s(1)}} \otimes \cdots \otimes X_{i_{s(p)}},$$

where $X_{i_k} \in \mathbf{g}$, $1 \le k \le p$ and S_p stands for the symmetric group of order p.

Let us write $[u]_n$ for the nth component of $u \in U(\mathbf{g})$ in the decomposition $U(\mathbf{g}) \simeq \bigoplus_{k=0}^{\infty} \sigma(S^k)$ [8].

Then for $P \in S^p$ and $Q \in S^q$ we define

$$P * Q = \sum_{k=0}^{\infty} (2\lambda)^k \sigma^{-1}([\sigma(P) \otimes \sigma(Q)]_{p+q-k}) := \sum_{k=0}^{\infty} \lambda^k C_k(P, Q). \tag{1}$$

Now, using linearity to extend the above expression to all polynomials, we define a $*$-product on $S(\mathbf{g})$. Moreover, the C_k in (1) are bi-differential operators on the space of polynomials, so it defines a differential $*$-product on $S(\mathbf{g})$, the so-called Gutt $*$-product.

Definition 3 [2]. A $*$-product on $S(\mathbf{g})$ will be called *global* if it is the restriction to $S(\mathbf{g})$ of a $*$-product on $C^\infty(\mathbf{g}^*)$.

A sufficient condition for a $*$-product on $S(\mathbf{g})$ to be global is that the C_k are bi-differential. Thus, the Gutt $*$-product is a differential $*$-product on $C^\infty(\mathbf{g}^*)$. From this point onwards, we shall use the summation convention on pairs of upper and lower indices, and for every smooth function f, $\partial^i f$ stands for $\partial f / \partial x_i$.

Let C_{ij}^k, $i, j, k = 1, \dots, n$ be the structure constants of a Lie algebra $\mathbf{g}$ relative to a basis $\{e_1, \dots, e_n\}$, and let also $(x_1, \dots, x_n)$ be a system of coordinates on $\mathbf{g}^*$ determined by the dual basis $\{e^1, \dots, e^n\}$. In these coordinates the Lie Poisson bracket of two smooth functions is given by

$$C_1(f, g)(x) = C_{ij}^k x_k \partial^i f \partial^j g,$$

and the 2-cochain C_2 of the Gutt $*$-product (1) is given by

$$C_2(f, g) = \tfrac{1}{2}\{x_i, x_j\}\{x_k, x_l\}\partial^{ik} f \partial^{jl} g - \tfrac{1}{3}\{x_k, \{x_i, x_j\}\}(\partial^{kj} f \partial^i g + \partial^i f \partial^{kj} g). \quad (2)$$

Proposition 3 [11]. *Let E be a differentiable 3-cocycle, null on the constants, on $C^\infty(\mathbf{g}^*)$. Then, if E is a 3-coboundary, one can choose a 2-cochain C such that $E = \partial C$ and*

$$C(f, g) = \sum_{0<p,q\le K} C_{i_1,\dots,i_p,j_1,\dots,j_q} \partial^{i_1,\dots,i_p} f \partial^{j_1,\dots,j_q} g,$$

where the coefficients $C_{i_1,\dots,i_p,j_1,\dots,j_q}$ are linear combinations of the coefficients $E_{k_1,\dots,k_a,l_1,\dots,l_b,m_1,\dots,m_c}$ of E.

Remark 1. Proposition 3 and the construction of the Gutt $*$-product in [11] yield that if an index i does not appear in any of the labels of the coefficients $E_{k_1,\dots,k_a,l_1,\dots,l_b,m_1,\dots,m_c}$ of the 3-cochain E_n defined by (3), then, it will not appear either in any of the coefficients $C_{i_1,\dots,i_p,j_1,\dots,j_q}$ of the corresponding 2-cochain C_n of the Gutt $*$-product.

Let $\mathbf{g}$ be a Lie algebra, and let $\mathbf{g}_x$ be the isotropy algebra of $x \in \mathbf{g}^*$. Let $e_1, \dots, e_n$, $e_{n+1}, \dots, e_m$ be a basis of $\mathbf{g}$ such that $e_1, \dots, e_n$ is a basis of $\mathbf{g}_x$, and as usual $e^1, \dots, e^n$, $e^{n+1}, \dots, e^m$ stand for the dual basis. From this point onwards, we will use greek indices to enumerate the basis of $\mathbf{g}_x$, i.e. $1 \le \alpha \le n$ and latin indices to enumerate the basis of the complement of $\mathbf{g}_x$ in $\mathbf{g}$, i.e. $n+1 \le i \le m$. A coordinate system $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ for $\mathbf{g}^*$ with respect to the above dual basis will be called a *T_x-coordinate system.*

Lemma 1. *Let O_x be a coadjoint orbit, and assume that $\mathbf{g}_x$ is an ideal. If $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ denotes a T_x-coordinate system for $\mathbf{g}^*$, then the non-vanishing coefficients $C_{i_1,\dots,i_p,j_1,\dots,j_q}$ of the 2-cochains (of the Gutt $*$-product)*

$$C_n(f, g) = \sum_{0<p,q\le K} C_{i_1,\dots,i_p,j_1,\dots,j_q}(y) \partial^{i_1,\dots,i_p} f \partial^{j_1,\dots,j_q} g, \quad y \in O_x, \quad n \ge 1,$$

are those whose indices satisfy, $n+1 \le i_1, \dots, i_p, j_1, \dots, j_q \le m$.

Proof. Let $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ be a T_x-coordinate system. Since $\mathbf{g}_x$ is an ideal, then $\mathbf{g}_x = \mathbf{g}_y$ $\forall y \in O_x$. Thus, we have

$$C_1(f, g)(y) = \{x_i, x_j\}(y) \partial^i f \partial^j g \quad \forall y \in O_x,$$

where $n+1 \le i, j \le m$.

Hence, the lemma is true for $k = 1$. Now suppose that it is true for C_k, $k \ge 1$.

Using Proposition 1 and the induction hypothesis, it follows that the non-vanishing coefficients of the 3-cochain

$$E_{k+1}(f,g,h) = \sum_{r+s=k,\ r,s\geq 1} (C_r(C_s(f,g),h) - C_r(f,C_s(g,h)))$$
$$= \sum_{0<a,b,c\leq K} E_{i_1,\ldots,i_a,j_1,\ldots,j_b,k_1,\ldots,k_c}\partial^{i_1,\ldots,i_a} f\partial^{j_1,\ldots,j_b} g\partial^{k_1,\ldots,k_c} h, \quad (3)$$

are those whose indices satisfy $n+1 \leq i_1,\ldots,i_a, j_1,\ldots,j_b, k_1,\ldots,k_c \leq m$.

Hence, by Proposition 3, we must have that the indices of the coefficients $C_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ of the 2-cochain C_{k+1} satisfy $n+1 \leq i_1,\ldots,i_p, j_1,\ldots,j_q \leq m\ \forall\ y \in O_x$. □

3. Tangential ∗-products

Throughout the rest of this exposition, the 2-cochains considered will be differential. Let $(\mathbf{M}, \Lambda)$ be a Poisson manifold, and let O be a symplectic leaf.

Definition 4. Let $x \in O$, a *differential operator* D on $\mathbf{M}$ is *tangential* to O at x, if there exist a neighborhood V of x in O and a neighborhood U of V in $\mathbf{M}$, such that when $\varphi_1, \varphi_2 \in C^\infty(U)$ with $\varphi_{1|_V} = \varphi_{2|_V}$, then

$$D(\varphi_1)_{|_V} = D(\varphi_2)_{|_V}.$$

A differential operator D on $\mathbf{M}$ is said to be tangential to O if it is tangential at x for all $x \in O$.

A *bi-differential* operator C on $\mathbf{M}$ is said to be *tangential* to O if, for any function $f \in C^\infty(\mathbf{M})$, the differential operators $C(f,\cdot), C(\cdot,f)$ are tangential to O.

Definition 5. A differential ∗-product is called tangential to O if all its cochains $C_n, n \geq 1$, are tangential.

Remark 2. Let C be a cochain of a tangential (to a given symplectic leaf O) ∗-product, and let f be a smooth function on $\mathbf{M}$ which is constant on an open subset V of O. Then Definition 4 implies that $C(f,g)_{|_V} = 0\ \forall g \in C^\infty(\mathbf{M})$.

Remark 3. Let $\mathbf{N}$ be a smooth submanifold (embedded) of a finite dimensional Euclidean space $\mathbf{E}$, and $p \in \mathbf{N}$. Then there exist independent smooth functions $f_1,\ldots,f_k$ on a neighborhood W of p in $\mathbf{E}$, such that

$$\mathbf{N} \cap W = \{x \in W \mid f_1(x) = 0,\ldots,f_k(x) = 0\},$$

where k is the codimension of $\mathbf{N}$ in $\mathbf{E}$.

In particular, Proposition 2 asserts that every symplectic leaf O is an immersed submanifold. Thus, $\forall y \in O$, there exists a neighborhood V of y in O such that V is an embedded submanifold of $(\mathbf{M}, \Lambda)$. Therefore, there exist independent functions $f_1, \ldots, f_k$ on a neighborhood W of y in $\mathbf{M}$ such that $V \cap W = \{x \in W \mid f_1(x) = 0, \ldots, f_k(x) = 0\}$, where k is the codimension of V in $\mathbf{M}$.

From this point onwards, by V_x we denote an open subset of O_x containing x.

Lemma 2. *Let O_x be a coadjoint orbit, and C_2 the second term of a differential and tangential (to O_x) $*$-product. If f is a smooth function such that $f_{|V_x}$ is constant, then*

$$C_2(fg, h)_{|V_x} = C_2(g, fh)_{|V_x} = fC_2(g, h)_{|V_x} \quad \forall g, h \in C^\infty(\mathbf{g}^*).$$

Proof. Since $\partial C_2(f, g, h) = fC_2(g, h) - C_2(fg, h) + C_2(f, gh) - C_2(f, g)h$, it follows that $fC_2(g, h)_{|V_x} - C_2(fg, h)_{|V_x} = C_1(g, C_1(h, f))_{|V_x} = 0$. □

4. Proof of the main theorem

Theorem 1. *Let $\mathbf{g}$ be a Lie algebra, and let O_x be a coadjoint orbit. Then the following are equivalent:*
(a) *the Gutt $*$-product is tangential to O_x;*
(b) *the 2-cochain C_2 of the Gutt $*$-product is tangential to O_x at x;*
(c) *$\mathbf{g}_x$ is an ideal.*

Proof. (a) ⇒ (b). By definition.

(b) ⇒ (c). Let $f \in C^\infty(\mathbf{g}^*)$ be such that $f_{|V_x}$ is constant, and $g, h \in C^\infty(\mathbf{g}^*)$. We define

$$R_2(f, g, h) = C_2(fg, h) - fC_2(g, h) - gC_2(f, h).$$

From Lemma 1, it is immediate that $R_2(f, g, h)_{|V_x} = 0$. If $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ is a T_x-coordinate system, then using (2) we obtain

$$R_2(f, g, h)(x) = -\tfrac{1}{3}\{x_k, \{x_\alpha, x_i\}\}(x)\partial^\alpha f \partial^k g \partial^i h, \quad x \in O_x.$$

That is, if C_2 is tangential to O_x at x, then

$$\{x_k, \{x_\alpha, x_i\}\}(x)\partial^\alpha f \partial^k g \partial^i h = 0 \quad \forall g, h \in C^\infty(\mathbf{g}^*).$$

In particular, for fixed k and i, we must have

$$\{x_k, \{x_\alpha, x_i\}\}(x)\partial^\alpha f(x) = 0$$

and using Remark 2 we conclude that

$$\{x_k, \{x_\alpha, x_i\}\}(x) = 0, \quad 1 \le \alpha \le n.$$

That is to say, $\langle x, [e_k, [e_\alpha, e_i]]\rangle = 0$, where k, i are fixed but arbitrary, so, $[e_\alpha, e_i] \in \mathbf{g}_x$, $1 \le \alpha \le n$, $n + 1 \le i \le m$. Hence, $\mathbf{g}_x$ is an ideal.

From Proposition 1 it is clear that $\partial^i f(x) = \partial^{jk} f(x) = 0$, $n + 1 \le i, j, k \le m$, so, from (2) it follows that $C_2(f, \cdot)(x) = 0$. Hence, C_2 is tangential to O_x at x.

(c) $\Rightarrow$ (a). Since $\mathbf{g}_x$ is an ideal, then, $\mathbf{g}_x = \mathbf{g}_y\ \forall y \in O_x$. Let $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ be a T_x-coordinate system, and let $f \in C^\infty(\mathbf{g}^*)$ such that $f_{|V_y}$ is constant. Then from Proposition 1 it follows that $\partial^{i_1, \ldots, i_p} f(y) = 0,\ \forall\ p \geq 1,\ \ n+1 \leq i_1, \ldots, i_p \leq m$. Therefore, from Lemma 1 it follows that $\forall g \in C^\infty(\mathbf{g}^*)$ and $k \geq 1$, $C_k(f, g)_{|V_y} = 0$. Hence, the Gutt $*$-product is tangential to O_x. □

Corollary 1. *Let* G *be a compact Lie group with Lie algebra* $\mathbf{g}$, *and let* O_x *be a coadjoint orbit in* $\mathbf{g}^*$. *Then the Gutt* $*$*-product is tangential to* O_x *if and only if* O_x *is* 0*-dimensional.*

Proof. Since O_x is a compact set in $\mathbf{g}^*$, then Proposition 1 asserts that $\mathbf{g}_x$ cannot be an ideal. □

Corollary 2. *Let* $\mathbf{g}$ *be a simple Lie algebra. Then the Gutt* $*$*-product is never tangential to any non-trivial orbit.*

4.1. Examples of tangential $*$-products

Example 1. Let $\mathbf{g}$ be the Lie algebra (*book algebra*) with basis (e_1, e_2, e_3) such that $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$. It is straightforward to check that for all $x \in \mathbf{g}^*$, $\mathbf{g}_x$ is an ideal. So, the Gutt $*$-product is tangential to all orbits.

Example 2. Let $\mathbf{g}$ be a *2-step nilpotent Lie algebra.* It is easy to check that the Gutt $*$-product is tangential to all the coadjoint orbits of $\mathbf{g}^*$. Furthermore, in [15], Richardson provides an example, for each positive integer $n > 1$, of an n-step, $(2n+1)$-dimensional nilpotent Lie algebra N_n for which all coadjoint orbits O_x satisfy Theorem 1(c). Thus, all these Lie algebras admit a tangential $*$-product on their duals.

Acknowledgements

The author would like to deeply thank J. Rawnsley for his invaluable help throughout this work, I. Sardis and Y. Terizakis for their helpful remarks, A. Loi and J.L. Cisneros for their help in the organization of the paper, and M. Cahen and S. Gutt for reading the manuscript.

References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin–Cummings, Menlo Park, CA, 1978).
[2] D. Arnal, M. Cahen and S. Gutt, Deformations on coadjoint orbits, J. Geom. Phys. 3 (1986) 327–351.
[3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and Quantization I and II, Ann. Phys. 111 (1978) 61–110 and 111–151.
[4] M. Cahen and S. Gutt, Regular $*$-representations of Lie algebras, Lett. Math. Phys. 6 (1982) 395–404.
[5] M. Cahen, S. Gutt and J.H. Rawnsley, On tangential star products for the coadjoint Poisson structure, Comm. Math. Phys., to appear.
[6] L. Corwin and F.P. Greenleaf, Representations of nilpotent Lie groups and their applications, Cambridge Studies in Advanced Mathematics (1990).

[7] M. De Wilde and P. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983) 487–496.
[8] J. Dixmier, *Enveloping Algebras* (North-Holland, Amsterdam, 1971).
[9] B.V. Fedosov, A simple geometrical construction of deformation quantization, J. Diff. Geom. 40 (1994) 213–238.
[10] M. Gerstenhaber, On the deformations of rings and algebras, Ann. of Math. 79 (1964) 59–103.
[11] S. Gutt, An explicit $*$-product on the cotangent bundle of a Lie group, Lett. Math. Phys. 7 (1983) 249–258.
[12] A. Masmoudi, Tangential formal deformations of the Poisson bracket and tangential star products on a regular Poisson manifold, J. Geom. Phys. 9 (1992) 155–177.
[13] H. Omori, Y. Maeda and A. Yoshioka, Weyl manifolds and deformation quantization, Adv. Math. 85 (1991) 224–255.
[14] H. Omori, Y. Maeda and Y. Yoshioka, Deformation quantization of Poisson algebras, Contemp. Math. 179 (1994) 213–240.
[15] L. Richardson, N-step nilpotent Lie groups with flat Kirillov orbits, Colloq. Math. 52 (1986) 191, 285–287.
[16] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557.